Physics 229A Gauge Theories: Homework Solution II

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Problem 1

In this problem we study the propagator for the most general (free) field theory of a massive vector boson that has, at most, two space-time derivatives. Therefore, we need to include all Lorentz invariant terms that satisfy these constraints. You can easily convince yourself that, after partial integrations and field redefinitions, the most general Lagrangian can be written as,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi^{-1}\partial^{\mu}A_{\nu}\partial^{\nu}A_{\mu} - \frac{1}{2}M^2 A_{\mu}A^{\mu} \ . \tag{1}$$

This is analogous to the gauged fixed Lagrangian of QED in the so-called R_ξ gauge (see Srednicki's book, Eq. (85.28)).

In momentum space, this Lagrangian takes the form,

$$\mathcal{L} = -\frac{1}{2}\tilde{A}(-k)\left[(k^2+M^2) + (1-\xi^{-1})k^{\mu}k^{\nu}\right]\tilde{A}(k) \ . \tag{2}$$

To find the propagator, we just need to invert the term in $[\cdots]$. This can be done just like in your QFT class. First write it in terms of the projection operator $P^{\mu\nu}(k) = g^{\mu\nu} - k^{\mu}k^{\nu}/k^2$ with $P_{\mu\nu}k^{\nu} = 0$. Then the inverse takes the form,

$$\Delta^{\mu\nu} = \frac{P^{\mu\nu}(k)}{k^2+M^2} + \frac{\xi k^{\mu}k^{\nu}/k^2}{k^2+\xi M^2} \ . \tag{3}$$

We can clearly see that the propagator has poles at $k^2 = -M^2$ and $k^2 = -\xi M^2$. The second pole requires $\xi > 0$. Otherwise we would be propagating a *Tachyon* (a particle with imaginary mass). The first pole is the usual contribution from the transverse modes of the gauge boson. The second pole comes from the longitudinal mode. Let us analyze this in a bit more detail.

If you go over the derivation of the usual (free) boson propagator, you can convince yourself that the momentum space propagator is related to a sum of oscillators:

$$\Delta^{\mu\nu}(k) = \sum_\lambda \frac{1}{k^2 + M_\lambda^2 - i\epsilon} \langle 0| a_\lambda^\mu (a_\lambda^\nu)^\dagger |0\rangle \ . \tag{4}$$

Therefore, if we look at each diagonal component, we see that the numerator is a positive number $|a_\lambda^\mu|0\rangle|^2$. Let us now look at the propagator (3). For the first term involving the transverse modes it is easy so see that in their rest frame ($k = (M, 0, 0, 0)$),

$$P^{00} = 0 \ , \quad P^{ii} = 1 \ . \tag{5}$$

This is consistent with three transverse modes. However, for the longitudinal mode we have,

$$\frac{\xi (k^0)^2}{k^2} = -\xi < 0 \ . \tag{6}$$

This is inconsistent with (4). Therefore we conclude that we must have $\xi = 0$. This sets the usual constraint $\partial_\mu A^\mu = 0$. This is why we chose the Proca Lagrangian in class.

Problem 2

In this problem we explore the emergence of the $U(1)_Y$ subgroup of the Standard Model electroweak symmetry group. As you discussed in class, Nature gave us a hint that, somehow, the symmetry group of the electroweak interactions should involve an $SU(2)$ (from the doublet structure, etc.). In fact, you saw that we have three conserved charges: Q, $Q^\pm$. At first sight they look like the candidates for the three generators of the $\mathbf{su(2)}$ Lie algebra. However, it turns out that their algebra is not closed. So we solve this by enlarging our group to $SU(2) \times U(1)_Y$. Let us now look for the generator of the extra $U(1)$[1].

[1] In this problem I am working with only one family of fermions. Adding the other families is straightforward and does not change the form of the $U(1)$ generator.

This generator (call it Y) must commute with all the other charges. First, we have that,

$$[Q^+, Q^-] = Q^3 \ , \tag{7}$$

where,

$$Q^- = \int d^3x \nu^\dagger P_L e \ , \quad Q^+ = (Q^-)^\dagger \ , \quad Q^3 = \frac{1}{2}\int d^3x \left[\nu^\dagger P_L \nu - e^\dagger P_L e\right] \ . \tag{8}$$

The other generator is,

$$Q_{em} = -\int d^3x e^\dagger e \ . \tag{9}$$

We can now use the equal-time commutator relation for a Dirac field,

$$\{\Psi_\alpha(x), \Psi^\dagger_\beta(y)\} = \delta_{\alpha\beta}\delta^3(x-y) \ , \quad \{\Psi_\alpha(x), \Psi_\beta(y)\} = 0 \ . \tag{10}$$

This follows directly from its Weyl components. It is easy to check that $[Q_{em}, Q^3] = 0$. To verify this, we just need to compute,

$$\begin{aligned}[e^\dagger(x)e(x), e^\dagger(y)P_L e(y)] &= [e^\dagger(x)e(x), e^\dagger_\alpha(y)](P_L e(y))_\alpha \\ &\quad + (e^\dagger(y)P_L)_\alpha [e^\dagger(x)e(x), e_\alpha(y)]\end{aligned} \tag{11}$$

Then using,

$$[e^\dagger(x)e(x), e^\dagger_\alpha(y)] = e^\dagger_\beta(x)e_\beta(x)e^\dagger_\alpha(y) - e_\alpha(y)e^\dagger_\beta(x)e^\dagger_\beta(x) = e^\dagger_\alpha(x)\delta^3(x-y) \ , \tag{12}$$

and its conjugate, it easy to show that the RHS of (11) vanishes.

The extra $U(1)$ must then be a linear combination of Q_{em} and Q^3. We now need to find the combination that commutes with $Q^\pm$. It is easy to show that $Y \sim Q_{em} - Q^3$ will do the trick. To show this, we can use (12) and its conjugate. Specifically, we have (suppressing the space-time indices and the obvious delta function)

$$[e^\dagger e, \nu^\dagger P_L e] = -\nu^\dagger P_L e \ , \tag{13}$$

and so

$$[Q_{em}, Q^-] = Q^- \ . \tag{14}$$

Moreover, one can show

$$[\nu^\dagger P_L \nu, \nu^\dagger_\alpha] = (\nu^\dagger P_L)_\alpha \ . \tag{15}$$

So,

$$
\begin{aligned}
[Q^3, Q^-] &= \frac{1}{2}\int d^3x d^3y \left([\nu^\dagger P_L \nu, \nu_\alpha^\dagger](P_L e)_\alpha - (\nu^\dagger P_L)_\alpha [e^\dagger P_L e, e_\alpha^\dagger]\right) \\
&= \int d^3x \nu^\dagger P_L e \\
&= Q^- \ . \qquad (16)
\end{aligned}
$$

Therefore, $[Q_{em} - Q^3, Q^\pm] = 0$.

Problem 3

This problem is (hopefully) a calculation that you did back in QFT: the muon decay rate. We can basically copy-paste most of it from Srednicki's lectures. Since we will be working in the approximation $M_W \gg m_\mu \gg m_e$, we can use the effective Fermi interaction,

$$\mathcal{L}_{\text{eff}} = 2\sqrt{2} G_F J_\mu^+ J^{\mu -} \ . \qquad (17)$$

The currents were written down in class, or you can find them in Srednicki Chapter 88. They are usually given in the one family approximation, so you just need to add the other families. For this problem we only need two families that we will call "e" and "μ" (you can also ignore the quarks). By expanding these currents it is easy to see that the muon will decay through the process: $\mu \to \nu_\mu \overline{\nu}_e e$.

I will not show all steps of the calculations since this is discussed in detail in chapter 88 of Srednicki's book. The first thing to do is to note that the term responsible for this decay process is,

$$\mathcal{L}_{\text{eff}} = 2\sqrt{2} G_F (\bar{e}\gamma^\rho P_L \nu_e)(\bar{\nu}_\mu \gamma_\rho P_L \mu) \ . \qquad (18)$$

Here μ is the dirac spinor corresponding to the muon. At this point, you can apply the Feynman rules for Dirac fields (e.g. chapter 45 of Srednicki's book). Let us label the momenta as in figure 88.1 of Srednicki's book. That is: $k_1 \sim \mu$, $k_1' \sim \nu_\mu$, $-k_2' \sim \bar{\nu}_e$ and $k_3' \sim e$.

Following the usual Feynan rules we get the amplitude,

$$\mathcal{T} = 2\sqrt{2} G_F (\bar{u}_{3'} \gamma^\rho P_L v_{2'})(\bar{u}_{1'} \gamma_\rho P_L u_1) \ . \qquad (19)$$

Now we can use the Fierz identities of section 36 to eliminate the gammas. Furthermore, we sum over final spins and average over initial spins. All of this makes heavy use of the "Spinor Technology" found in Srednicki's book. The final result is pretty simple,

$$\langle|\mathcal{T}|^2\rangle = 64G_F^2(k_1 \cdot k_2')(k_1' \cdot p_3') \ . \tag{20}$$

In the rest frame of the muon, the decay rate can be found from Eq. (11.48) of the book but with $E_1 = m_\mu$. That is,

$$\Gamma = \frac{32G_F^2}{m_\mu}\int(p_1 \cdot p_2')(p_1' \cdot p_3')d\text{LIPS}_3(p_1) \ . \tag{21}$$

What follows now is basically problem 11.3 of the book. Luckily (for me) I can recycle the solution from last quarter 221 A. Let me mention that, from now on, I will make the approximation of $m_\mu \gg m_e$ so that I will take the electron to be massless.

Let us divide the problem in various parts:

a) First, we can relabel,

$$dLIPS_3(k_1) = (2\pi)^4\delta^4(k_1 - k_1' - k_2' - k_3')d\tilde{k}_1'd\tilde{k}_2'd\tilde{k}_3' = d\tilde{k}_3'dLIPS_2(k_1 - k_3') \ . \tag{22}$$

The decay rate can be written as,

$$\Gamma = \frac{32G_F^2}{m_\mu}\int \tilde{k}_3' k_{1\mu}k_{3\nu}'\int k'^{\mu}_2 k'^{\nu}_1 d\text{LIPS}_3(k_1 - k_3') \ . \tag{23}$$

b) Let us now look at the integral ($k \equiv k_1 - k_3'$),

$$\int (k_1')^\mu(k_2')^\nu dLIPS_2(k) \equiv F^{\mu\nu}(k) \ . \tag{24}$$

Since the integral only depends on the vector k, and has two indices, the only Lorentz invariant objects that we can construct are: $g^{\mu\nu}$ and $k^\mu k^\nu$. These are the only terms that can appear here. By dimensional analysis, one has $[F] = [k^2]$. Thus,

$$F^{\mu\nu}(k) = Ak^2g^{\mu\nu} + Bk^\mu k^\nu \ , \tag{25}$$

where A, B are constant.

c) Let us calculate,

$$\begin{aligned}\int dLIPS_2(k) &= (2\pi)^4 \int_{q_1^2=0,\ q_2^2=0} \tilde{dq_1}\tilde{dq_2}\delta(k^0-(q_1)^0-(q_2)^0)\delta^3(\vec{k}-\vec{q_1}-\vec{q_2}) \\ &= \pi \int_{q_1^2=0} \frac{\tilde{dq_1}}{|\vec{k}-\vec{q_1}|}\delta(k^0-|\vec{q_1}|-|\vec{k}-\vec{q_1}|)\ . \end{aligned} \tag{26}$$

Since k is a time-like vector, we can go to a frame where $\vec{k}=0$. Then the integral gives,

$$\int dLIPS_2(k) = \frac{\pi}{2(2\pi)^3}(4\pi)\int_0^\infty dr\delta(k^0-2r) = \frac{1}{8\pi}\ . \tag{27}$$

Here, I have used the fact that $\delta(ax)=\frac{1}{a}\delta(x)$.

d) Let us now contract (24) with the metric. We get,

$$\begin{aligned} k^2(4A+B) &= \int (k_1'\cdot k_2')dLIPS_2(k) \\ &= \frac{1}{2}\int (k_1'+k_2')^2 dLIPS_2(k) \\ &= \frac{1}{2}\int k^2 dLIPS_2(k)\ , \end{aligned} \tag{28}$$

where I have used momentum conservation in the last line. Moreover, remember that k_1', k_2' are null. Thus,

$$4A+B = \frac{1}{16\pi}\ . \tag{29}$$

Contracting (24) with k itself we get,

$$\begin{aligned} (k^2)^2(A+B) &= \int (k\cdot k_1')(k\cdot k_2')dLIPS_2(k) \\ &= \int ((k_1'+k_2')\cdot k_1')((k_1'+k_2')\cdot k_2')dLIPS_2(k) \\ &= \int (k_2'\cdot k_1')^2 dLIPS_2(k) \\ &= \frac{1}{4}\int ((k_1'+k_2')^2)^2 dLIPS_2(k) \\ &= \frac{1}{4}\int (k^2)^2 dLIPS_2(k)\ . \end{aligned} \tag{30}$$

We find,

$$A + B = \frac{1}{32\pi} \ . \tag{31}$$

Thus,

$$A = \frac{1}{96\pi} \ , \quad B = \frac{1}{48\pi} \ . \tag{32}$$

e) We can now finally calculate the decay rate of the muon. From the results above, and (23) we get,

$$\Gamma = \frac{32 G_F}{m_\mu} \int d\tilde{k}'_3 [A k^2 (k_1 \cdot k'_3) + B (k \cdot k_1)(k \cdot k'_3)] \ . \tag{33}$$

If we work on the rest frame of the muon, and we assume that the mass of the electron is negligible, we get,

$$k_1 = (m_\mu, \vec{0}) \ \text{(muon)} \ , \quad k'_3 \equiv (E_e, E_e \hat{q}) \ \text{(electron)} \ , \tag{34}$$

where $\hat{q}^2 = 1$.

In this frame, one can write the integrand of Γ as,

$$[A k^2 (k_1 \cdot k'_3) + B (k \cdot k_1)(k \cdot k'_3)] = m_\mu^2 E_e [A(m_\mu - 2E_e) + B(m_\mu - E_e)] \ . \tag{35}$$

The integral measure in Γ can be written as,

$$d\tilde{k}'_3 = \frac{d^3 k'_3}{2(2\pi)^3 E_e} = \frac{1}{2(2\pi)^3} d\Omega dE_e E_e \ , \tag{36}$$

where $d\Omega$ it the solid angle in the direction of $\hat{q}$.

Thus, we can finally write,

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2 E_e^2 m_\mu}{4\pi^3} \left(m_\mu - \frac{4}{3} E_e \right) \ . \tag{37}$$

Note that the maximum allowed value of E_e is reached when the electron is emitted in one direction and the two neutrinos in the opposite direction. Using momentum conservation we get,

$$(E_e)^2_{\max} = |\vec{k'_3}|^2 = |\vec{k'_1} + \vec{k'_2}|^2 = |\vec{k'_1}|^2 + |\vec{k'_2}|^2 + 2|\vec{k'_1}||\vec{k'_2}| = (|\vec{k'_1}| + |\vec{k'_2}|)^2 \ . \tag{38}$$

Energy conservation then tells us that,

$$(|\vec{k'_1}| + |\vec{k'_2}|) = m_\mu - (E_e)_{\max} \ . \tag{39}$$

Thus we get,

$$(E_e)_{\max} = \frac{m_\mu}{2} \ . \tag{40}$$

f) It is now straightforward to integrate (37) from $E_e = 0$ to $E_e = m_\mu/2$ to get,

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3} \ . \tag{41}$$

Problem 3

For this problem we want to calculate how many independent phases do we get in the KM matrix for n generations of fermion doublets. Let us label the *gauge eigenstates* with the notation,

$$\begin{pmatrix} u'_A \\ d'_A \end{pmatrix} \tag{42}$$

where the index $A = 1, \ldots, n$ labels the families. These are all left handed fields. To diagonalize the mass matrix we make the redefinition (with a similar one for the right-handed fields),

$$u'_A = (U_{(u)})_{AB} u_B \ , \quad d'_A = (U_{(d)})_{AB} d_B \ . \tag{43}$$

Now the unprimed fields are the mass eigenstates.

This shift introduces the following *unitary* matrix to the charged currents:

$$J_\mu^+ = \bar{u}'_A \gamma_\mu d'_A = \bar{u}_A \gamma_\mu S_{AB} u_B \ , \tag{44}$$

where

$$S_{AB} = (U_{(u)}^\dagger U_{(d)})_{AB} \ . \tag{45}$$

Note that S is a unitary matrix. To calculate the number of independent parameters of S, we recall that a general complex matrix has $2n^2$. Consider now the condition $S^\dagger S = 1$. The LHS is a Hermitean matrix: $(S^\dagger S)^\dagger = S^\dagger S$. It is very simple to count the d.o.f. of a Hermiten matrix: $2n^2 - \frac{1}{2}(2n^2 - 2n) - n = n^2$. In other words, only the real diagonal and the complex upper-triangular parts. This is also the number of constraints that we put using the unitarity condition. Therefore, the remaining number of d.o.f. of S are: $2n^2 - n^2 = n^2$.

We can now write the charge eigenstates in terms of the mass eigenstates as,

$$\begin{pmatrix} u''_A \\ d''_A \end{pmatrix} = \begin{pmatrix} u_A \\ S_{AB} d_B \end{pmatrix} . \tag{46}$$

Since an overall phase for the doublets is immaterial, we can eliminate the phase of (say) S_{11} by redefining u_1:

$$\begin{pmatrix} u_1 \\ R_{11} e^{i\phi_{11}} d_1 + S_{12} d_2 + \cdots \end{pmatrix} \to e^{i\phi_{11}} \begin{pmatrix} u_1 \\ R_{11} d_1 + S_{12} e^{-i\phi_{11}} d_2 + \cdots \end{pmatrix} . \tag{47}$$

We can then redefine $d_2, \ldots, d_n$ to absorb all phases of this doublet, giving,

$$e^{i\phi_{11}} \begin{pmatrix} u_1 \\ R_{1A} d_A \end{pmatrix} . \tag{48}$$

Doing this we eliminate n phases. Finally, we can redefine $u_2, \ldots, u_n$ to absorb one more phase of the other doublets (just like we did in (47)). This gets rid of $n-1$ extra phases. In total we killed $2n-1$ phases.

To calculate the remaining number of phases in S, we need to subtract the number of real parameters. Since the differences between a unitary matrix and an orthogonal matrix are precisely the phases, we can just subtract the number of free parameters of a orthogonal matrix: $n(n-1)/2$. Therefore, the total number of free angles in S is,

$$n^2 - (2n-1) - \frac{1}{2} n(n-1) = \frac{1}{2}(n-1)(n-2) . \tag{49}$$

Now consider a theory with right handed charged currents. Let the right handed Weyl gauge eigenstates be $\bar{u}'_A$ and $\bar{d}'_A$ (The bar is not a conjugation!). Now recall that the mass matrix is actually diagonalized by by two unitary transformations:

$$m^{(u)}_{AB} \bar{u}'^{\dagger}_A u'_B + m^{(d)}_{AB} \bar{d}'^{\dagger}_A d'_B \to m^{(u)}_{AA} \bar{u}^{\dagger}_A u_A + m^{(d)}_{AA} \bar{d}^{\dagger}_A d_A , \tag{50}$$

with

$$u'_A = (U_{(u)})_{AB} u_B , \quad \bar{u}'_A = (\bar{U}_{(u)})_{AB} \bar{u}_B , \quad \text{etc.} \tag{51}$$

You can follow the procedure above and see that we will have the following charge eigenstates,

$$\begin{pmatrix} \bar{u}''_A \\ \bar{d}''_A \end{pmatrix} = \begin{pmatrix} \bar{u}_A \\ \tilde{S}_{AB} \bar{d}_B \end{pmatrix} , \tag{52}$$

with $\tilde{S} = \bar{U}^\dagger_{(u)} \bar{U}_{(d)}$. However, we can't really absorb any more phases because they would change the mass terms (50) and all phases of the left handed fields have been already set. So we are stuck will the $n^2 - \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$ phases of $\tilde{S}$.

Therefore, adding right-hand charged currents will led to the following number of total phases:

$$\frac{1}{2}(n-1)(n-2) + \frac{1}{2}n(n+1) = n^2 - n + 1 \ . \tag{53}$$